Journal of Statistical Physics, Vol. 45, Nos. 5/6, 1986

On the Theory of Brownian Motion with the Alder–Wainwright Effect

Yasunori Okabe[1]

Received October 22, 1985; revision received July 29, 1986

The Stokes–Boussinesq–Langevin equation, which describes the time evolution of Brownian motion with the Alder–Wainwright effect, can be treated in the framework of the theory of KMO-Langevin equations which describe the time evolution of a real, stationary Gaussian process with T-positivity (reflection positivity) originating in axiomatic quantum field theory. After proving the fluctuation-dissipation theorems for KMO-Langevin equations, we obtain an explicit formula for the deviation from the classical Einstein relation that occurs in the Stokes–Boussinesq–Langevin equation with a white noise as its random force. We are interested in whether or not it can be measured experimentally.

KEY WORDS: Stokes–Boussinesq–Langevin equation; Alder–Wainwright effect; white noise; Kubo noise; diffusion constant; Einstein relation.

1. INTRODUCTION

About 15 years ago, Alder and Wainwright[1,2] discovered a long-time tail behavior ($\propto t^{-3/2}$) of the velocity autocorrelation function for hard spheres by a computer simulation. Since then, much effort has gone into confirming such an *Alder–Wainwright effect* in both experiment and theory on the basis of Kubo's linear response theory in statistical physics.[3,4,9,11,17,26,27,30,31] In the course of these investigations, it has become clear that the Brownian motion with the Alder–Wainwright effect can be described by an equation treated by Stokes[28] and Boussinesq[5,6] in hydrodynamics. Its equation with a random force reads

$$m^* \frac{dX(t)}{dt} = -6\pi r\eta X(t) - 6\pi r^2 \left(\frac{\rho\eta}{\pi}\right)^{1/2} \int_{-\infty}^{t} \frac{1}{(t-s)^{1/2}} \frac{dX(s)}{ds}\, ds + W(t) \qquad (1.1)$$

[1] Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo, Hakkaido, 060, Japan.

0022-4715/86/1200-0953$05.00/0

where m^* is an effective mass given by

$$m^* = m + \tfrac{2}{3}\pi r^3 \rho \tag{1.2}$$

Here we consider the motion of a sphere of radius r and mass m moving with an arbitrary velocity $X(t)$ at time t in a fluid with viscosity η and density ρ subject to a random force $W(t)$ at time t.

On the other hand, we know[7,29] that the time evolution of the Ornstein–Uhlenbeck Brownian motion as treated by Einstein[8] and Langevin[19] reads

$$m \frac{dX(t)}{dt} = -6\pi r\eta X(t) + [2kT(6\pi r\eta)]^{1/2} \dot{B}(t) \tag{1.3}$$

where k is the Boltzmann constant, T is the temperature of the fluid, and $\dot{B}$ is a Gaussian white noise, that is, the time derivative of a standard Brownian motion B. It is known[7,29] that the Ornstein–Uhlenbeck Brownian motion can be characterized as a stationary Gaussian process with Markovian property.

We raise the following questions:

1. What kind of qualitative nature does the solution of Eq. (1.1) have?

2. Conversely, can a stochastic process with such a qualitative nature be governed by Eq. (1.1)?

For that purpose, we have to clarify the true character of the random force W in Eq. (1.1) from the viewpoint of the theory of stochastic processes.

In a series of papers[21–24] we have tried to characterize the class of stochastic differential equations describing the time evolution of stationary Gaussian processes with T-positivity. In Ref. 24 we obtained two kinds of equations, called the *first KMO-Langevin equation* and the *second KMO-Langevin equation*. The former was derived from the structure theorem of the outer function and in consequence it has a *white noise* as a random force, which gives a generalization and a refinement of the $[\alpha, \beta, \gamma]$-Langevin equation in Refs. 22 and 23. The latter was derived from the structure theorem of the Laplace–Fourier transform of the correlation function, and then the random force in it is in general colored, which we called the *Kubo noise* with its physical origin in Kubo's linear response theory in statistical physics.[13–16]

Our motivation in these investigations was to clarify the mathematical structure of the fluctuation-dissipation theorem in Kubo's linear response theory in statistical physics with the desire for application from mathemtics to physics.

In the first part of this paper, we will show in Sections 2 and 3 a *generalized fluctuation-dissipation theorem* on the basis of the first KMO-

Langevin equation, which gives a generalization and a refinement of the results in Refs. 22 and 23. We further show in Section 4 *Kubo's fluctuation-dissipation theorem* on the basis of the second KMO-Langevin equation. The physical meaning and significance of the fluctuation-dissipation theorems will be given in Section 2.

As a consequence, we will find (Theorems 2.1, 3.1, and 4.1) that a fundamental difference between two kinds of fluctuation-dissipation theorems stated above lies in the fact that the classical *Einstein relation* holds on the basis of the second KMO-Langevin equation, but does not hold on the basis of the first KMO-Langevin equation. Its degree of deviation from the Einstein relation can be calculated. We note that the description of the fluctuation-dissipation theorem should be given on the basis of the equation that describes the time evolution of the process under consideration, because under additional conditions we can rewrite the first (resp. second) KMO-Langevin equation into the second (resp. first) KMO-Langevin equation by way of a change of the coefficients in the equations.

In the last part of this paper, we will find in Sections 5 and 6 that the correlation function R_{K} in the investigations of Hauge and Martin-Löf[11] and Kubo,[17] whose object is to confirm the Alder–Wainwright effect, has a qualitative nature of T-positivity. We then see from Section 4 that R_{K} is realized as the autocorrelation function of the unique stationary solution X_{K} for the Stokes–Boussinesq–Langevin equation (1.1) where the random force W is Kubo noise, as a concrete example of the second KMO-Langevin equation. We note that R_{K} cannot be realized as an autocorrelation function of the stationary solution for some first KMO-Langevin equation.

We know[24] that the white noise plays the same role that the Kubo noise does as a random force in the derivation of KMO-Langevin equations. We will find in Section 7 that the autocorrelation function R_W of the unique stationary solution X_W for the Stokes–Boussinesq–Langevin equation (1.1) with the white noise $\alpha_W \dot{B}$ as random force has the same qualitative nature of T-positivity as R_{K}, where α_W is a positive constant. We note that X_W can be a unique stationary solution for some second KMO-Langevin equation whose systematic part is different from the one in equation (1.1). Furthermore, we will show that R_W satisfies the *Alder–Wainwright effect*, similarly as R_{K}:

$$\lim_{t \to \infty} (\beta_{\mathrm{SB}}\, t)^{3/2} R_{\mathrm{K}}(t) = \frac{R_{\mathrm{K}}(0)}{2\sqrt{\pi}}\, a \tag{1.4}$$

$$\lim_{t \to \infty} (\beta_{\mathrm{SB}}\, t)^{3/2} R_{\mathrm{W}}(t) = \frac{\sqrt{\pi}\, R_W(0)}{2} \left(\int_0^\infty \frac{1}{1 + y + a\sqrt{y}} \frac{\sqrt{y}}{(1-y)^2 + a^2 y}\, dy \right)^{-1} \tag{1.5}$$

where

$$\beta_{\mathrm{SB}} = 6\pi r\eta/m^* \tag{1.6}$$

$$a = (6\pi r^3\rho/m^*)^{1/2} \tag{1.7}$$

By applying the *Einstein relation* in Section 4 and the *generalized Einstein relation* in Section 2 to X_{K} and X_W, we will obtain in Sections 6 and 7 the following results:

$$D_{\mathrm{K}} = \frac{R_{\mathrm{K}}(0)}{\beta_{\mathrm{SB}}} \tag{1.8}$$

$$D_W = \frac{R_W(0)}{\beta_{\mathrm{SB}}}\left(1 + a\,\frac{\int_0^\infty (1+y+a\sqrt{y})^{-1}[(1-y)^2+a^2y]^{-1}\,dy}{(1+y+a\sqrt{y})^{-1}\sqrt{y}/[(1-y)^2+a^2y]\,dy}\right) \tag{1.9}$$

where D_{K} and D_W are the diffusion constants of X_{K} and X_W, respectively, defined by

$$D_{\mathrm{K}} = \int_0^\infty R_{\mathrm{K}}(t)\,dt \tag{1.10}$$

$$D_W = \int_0^\infty R_W(t)\,dt \tag{1.11}$$

We note that D_{K} and D_W are concretely obtained as follows:

$$D_{\mathrm{K}} = 1/3(2\pi)^{1/2} r\eta \tag{1.12}$$

$$D_W = \tfrac{1}{2}(\alpha_W/6\pi r\eta)^2 \tag{1.13}$$

In Section 8 we will investigate the $\lim a \to 0$ behavior of the deviation from the Einstein relation in (1.9) and the processes X_{K} and X_W. We note that for a fixed η the $\lim a \to 0$ behavior is equivalent to the $\lim \rho \to 0$ behavior by (1.2) and (1.7). We will show that

$$\lim_{a\to 0} D_W\left(\frac{R_W(0)}{\beta_{\mathrm{SB}}}\right)^{-1} = 1 \tag{1.14}$$

and under the condition

$$\alpha_W = (12\sqrt{2}\pi^{3/2} r\eta)^{1/2} \tag{1.15}$$

we have

$$\lim_{a\to 0} X_{\mathrm{K}} = \lim_{a\to 0} X_W = X_\infty \tag{1.16}$$

where X_∞ is the Ornstein–Uhlenbeck Brownian motion whose time evolution is governed by Eq. (1.3) for the case where $kT = (2\pi)^{1/2}$. We note that under the condition (1.15), D_{K} and D_W coincide with the diffusion constant of X_∞. Furthermore, we will show the following interesting limit theorem under the condition (1.15):

$$\lim_{a \to 0} \left(\lim_{t \to \infty} \frac{R_W(t)}{R_{\mathrm{K}}(t)} \right) = 2 \tag{1.17}$$

$$\lim_{t \to \infty} \left(\lim_{a \to 0} \frac{R_W(t)}{R_{\mathrm{K}}(t)} \right) = 1 \tag{1.18}$$

In closing this section we are interested in whether the deviation from the Einstein relation in (1.9) that occurs in the case where the random force W in the Stokes–Boussinesq–Langevin equation (1.1) is a white noise can be measured experimentally. In a forthcoming paper[25] we will find that for the discrete time series the Einstein relation deviates from the one in the Markovian case not only on the basis of the first KMO-Langevin equation, but also on the basis of the second KMO-Langevin equation. Therefore, it seems that such a criterion as entropy is needed, besides the Einstein relation, in order to determine which of the white noise and the Kubo noise is adequate as the random force in the KMO-Langevin equation under the condition that its systematic part is given for modeling. The entropy criterion will be discussed in Ref. 25.

2. A GENERALIZED FLUCTUATION-DISSIPATION THEOREM. 1

Let $X = (X(t);\, t \in \mathbf{R})$ be a real, stationary Gaussian process with mean zero and covariance function R of the form

$$R(t) = \int_0^\infty e^{-|t|\lambda}\, \sigma(d\lambda) \qquad (t \in \mathbf{R}) \tag{2.1}$$

where σ is a Borel measure on $[0, \infty)$ satisfying the following condition:

$$\sigma(\{0\}) = 0 \qquad \text{and} \qquad 0 < \sigma([0, \infty)) < \infty \tag{2.2}$$

By Theorem 2.1 in Ref. 21, we know that R has a spectral density Δ of Hardy type such that

$$R(t) = \int_{\mathbf{R}} e^{-it\xi}\, \Delta(\xi)\, d\xi \qquad (t \in \mathbf{R}) \tag{2.3}$$

$$\Delta(\xi)=\frac{1}{\pi}\int_0^\infty \frac{\lambda}{\xi^2+\lambda^2}\,\sigma(d\lambda) \qquad (\xi\in\mathbf{R}-\{0\}) \tag{2.4}$$

$$\frac{\log \Delta(\xi)}{1+\xi^2}\in L^1(\mathbf{R}) \tag{2.5}$$

Then we can obtain an outer function h and a canonical representation kernel E with a one-dimensional Brownian motion $(B(t);\, t\in\mathbf{R})$ having the properties (H.1)–(H.4), (E.1)–(E.4), (B.1), and (B.2) in Ref. 24.

In this section, we shall treat the case where the following conditions are satisfied:

$$\int_0^\infty \lambda^{-1}\,\sigma(d\lambda)<\infty \tag{2.6}$$

$$\int_0^\infty \lambda\sigma(d\lambda)<\infty \tag{2.7}$$

By Theorems 2.2 and 3.1 in Ref. 24, we obtain *first KMO-Langevin data* $(\alpha, \beta, \rho)\in\mathscr{L}_1$ associated with σ such that for any $\zeta\in\mathbf{C}^+$

$$h(\zeta)=\frac{\alpha}{(2\pi)^{1/2}}\frac{1}{\beta-i\zeta-i\zeta\int_0^\infty [1/(\lambda-i\zeta)]\,\rho(d\lambda)} \tag{2.8}$$

We recall that $(\alpha, \beta, \rho)\in\mathscr{L}_1$ means that

$$\alpha>0 \qquad \text{and} \qquad \beta>0 \tag{2.9}$$

$$\begin{aligned}&\rho \text{ is a Borel measure on } [0,\infty) \text{ with } \rho(\{0\})=0\\ &\text{and } \int_0^\infty \frac{1}{\lambda+1}\,\rho(d\lambda)<\infty\end{aligned} \tag{2.10}$$

Furthermore, we know from Theorem 4.1 in Ref. 24 that the time evolution of the process X is governed by the following *first KMO-Langevin equation*:

$$\dot{\mathbf{X}}=-\beta\mathbf{X}-\lim_{\varepsilon\downarrow 0}\gamma_\varepsilon * \dot{\mathbf{X}}+\alpha\dot{\mathbf{B}} \tag{2.11}$$

where for each $\varepsilon>0$, γ_ε is a function on $\mathbf{R}$ defined by

$$\gamma_\varepsilon(t)=\chi_{(0,\infty)}(t)\int_\varepsilon^\infty e^{-t\lambda}\,\rho(d\lambda) \tag{2.12}$$

We note that Eq. (2.11) holds in the sense of random tempered distributions. Then we shall show the following:

Theorem 2.1.:

(i) For any $\zeta \in \mathbf{C}^+ \cup (\mathbf{R} - \{0\})$,

$$\frac{1}{\beta - i\zeta - i\zeta \lim_{\varepsilon \downarrow 0} \int_0^\infty e^{i\zeta t} \gamma_\varepsilon(t)\, dt} = \frac{h(\zeta)}{\int_{\mathbf{R}} \operatorname{Re} h(\xi + i0)\, d\xi}$$

(ii) $\alpha^2/2 = R(0) C_{\beta,\gamma}$

where $C_{\beta,\gamma}$ is a positive constant defined by

$$C_{\beta,\gamma} = \pi \left\{ \int_{\mathbf{R}} \left| \beta - i\xi \left[1 + \lim_{\varepsilon \downarrow 0} \int_0^\infty e^{i\xi t} \gamma_\varepsilon(t)\, dt \right] \right|^{-2} d\xi \right\}^{-1} \tag{2.13}$$

(iii) $D = \alpha^2/2\beta^2$

where D is a positive constant defined by

$$D = \int_0^\infty R(t)\, dt \tag{2.14}$$

(iv) $D = \dfrac{R(0)}{\beta} \dfrac{C_{\beta,\gamma}}{\beta}$

(v) $C_{\beta,\gamma} - 1 = \displaystyle\int_0^\infty \frac{R(t)}{R(0)} \gamma(t)\, dt$

Proof. By (E.2) and Lemma 2.7 in Ref. 24, we have

$$\alpha = (2/\pi)^{1/2} \int_{\mathbf{R}} \operatorname{Re} h(\xi + i0)\, d\xi \tag{2.15}$$

which together with (2.8) yields (i). Similar to (9.13) in Ref. 22, we get (ii) from (H.4) in Ref. 24 and (2.8). Since

$$D = \int_0^\infty \lambda^{-1} \sigma(d\lambda) \tag{2.16}$$

then (iii) follows from formulas (i) and (ii) in Theorem 3.3 in Ref. 24. From (ii) and (iii), we immediately have (iv). Part (v) can be shown as follows. First, we claim that for any $t > 0$,

$$E(t) = (2\pi)^{1/2}\alpha - \beta \int_0^t E(s)\, ds - (\gamma * E)(t) \tag{2.17}$$

where γ is a function on $\mathbf{R}$ defined by

$$\gamma(t) = \chi_{(0,\infty)}(t) \int_0^\infty e^{-t\lambda} \rho(d\lambda) \tag{2.18}$$

By Lemma 4.3 in Ref. 24, which is equivalent to Eq. (2.11), we have

$$X(t)-X(0)=\alpha(B(t)-B(0))-\int_0^t X(s)\,ds$$
$$-\lim_{\varepsilon\downarrow 0}[(\gamma_\varepsilon * X)(t)-(\gamma_\varepsilon * X)(0)]$$

for any $t>0$. By multiplying both sides by $B(t)-B(0)$ and then noting (B.1), (B.2), Lemma 2.8(iv), and (4.11) in Ref. 24,

$$\int_0^t E(s)\,ds=(2\pi)^{1/2}\alpha t-\beta\int_0^t\left[\int_0^s E(u)\,du\right]ds-\int_0^t(\gamma * E)(s)\,ds$$

and then by differentiating both sides with respect to t, we have (2.17). Next we claim that for any $t>0$,

$$R(t)=\frac{\alpha^2}{2\beta}-\beta\int_0^t R(s)\,ds-\int_0^\infty R(t-s)\,\gamma(s)\,ds \tag{2.19}$$

By (E.4) in Ref. 24 and (2.17), we have, for any $t>0$,

$$R(t)=\frac{\alpha}{(2\pi)^{1/2}}\int_0^\infty E(s)\,ds-\frac{\beta}{2\pi}\int_0^\infty\left[\int_0^{t+s}E(u)\,du\right]E(s)\,ds$$
$$-\frac{1}{2\pi}\int_0^\infty(\gamma * E)(t+s)\,E(s)\,ds$$

Furthermore, we can see from (E.4) and Theorem 2.1(iii) in Ref. 24, part (iii) of Theorem 2.1 in this paper, and (2.16) that for any $t>0$,

$$\frac{1}{2\pi}\int_0^\infty\left[\int_0^{t+s}E(u)\,du\right]E(s)\,ds=\frac{\alpha^2}{2\beta^2}+\int_0^t R(s)\,ds$$

Therefore, we get

$$R(t)=\frac{\alpha^2}{2\beta}-\beta\int_0^t R(s)\,ds-\frac{1}{2\pi}\int_0^\infty(\gamma * E)(t+s)\,E(s)\,ds$$

Since by (E.4) in Ref. 24 we have, for any $t>0$,

$$\frac{1}{2\pi}\int_0^\infty(\gamma * E)(t+s)\,E(s)\,ds=\int_0^\infty R(t-s)\,\gamma(s)\,ds$$

we can obtain (2.19).

Finally, we separate

$$\int_0^\infty R(t-s)\,\gamma(s)\,ds = \int_0^t R(t-s)\,\gamma(s)\,ds + \int_t^\infty R(t-s)\,\gamma(s)\,ds$$
$$= \mathrm{I}(t) + \mathrm{II}(t)$$

By Lemma 2.8(iv) in Ref. 24,

$$\lim_{t \downarrow 0} \mathrm{I}(t) = 0$$

By the monotone convergence theorem,

$$\lim_{t \downarrow 0} \mathrm{II}(t) = \int_0^\infty R(s)\,\gamma(s)\,ds$$

Therefore, we see from (2.19) that

$$R(0) = \frac{\alpha^2}{2\beta} - \int_0^\infty R(s)\,\gamma(s)\,ds$$

By combining it with (iv) in Theorem 2.1, we have (v). ∎

For future use, we arrange (2.17) and (2.19) into the following:

Corollary 2.1. For any $t > 0$,

(i) $$E(t) = (2\pi)^{1/2} - \beta \int_0^t E(s)\,ds - (\gamma * E)(t)$$

(ii) $$R(t) = \frac{\alpha^2}{2\beta} - \beta \int_0^t R(s)\,ds - \int_0^\infty R(t-s)\,\gamma(s)\,ds$$

By taking the same argument as Theorem 9.1 in Ref. 21 and then using Theorem 2.1(v), we have

Theorem 2.2:

(i) $C_{\beta,\gamma}/\beta \geqslant 1$.

(ii) The following five statements are equivalent:

(a) $C_{\beta,\gamma}/\beta = 1$.

(b) $\gamma = 0$.

(c) $\rho = 0$.

(d) $R = cE$ in $(0, \infty)$ with a positive constant c.

(e) X has a simple Markovian property.

Before stating the physical meaning of Theorem 2.1, we give the following example:

Example 2.1. Let $X = (X(t);\ t \in \mathbf{R})$ be an Ornstein–Uhlenbeck Brownian motion whose time evolution is governed by the $[\alpha, \beta, 0]$-Langevin equation[7,22]

$$\dot{X} = -\beta X + \alpha \dot{B} \tag{2.20}$$

where α and β are positive constants and $(B(t);\ t \in \mathbf{R})$ is a one-dimensional Brownian motion. Then it can be seen that the covariance function $R_{\alpha,\beta}$, outer function $h_{\alpha,\beta}$, canonical representation kernel $E_{\alpha,\beta}$, and positive constants $C_{\beta,0}$ and $D_{\alpha,\beta}$, which are defined by (2.13) and (2.14), are given by

$$R_{\alpha,\beta}(t) = \frac{\alpha^2}{2\beta} e^{-\beta|t|} \qquad (t \in \mathbf{R}) \tag{2.21}$$

$$h_{\alpha,\beta}(\zeta) = \frac{\alpha}{2\pi} \frac{1}{\beta - i\zeta} \qquad (\zeta \in \mathbf{C}^+) \tag{2.22}$$

$$E_{\alpha,\beta}(t) = \chi_{[0,\infty)}(t)(2\pi)^{1/2} \alpha e^{-\beta t} \qquad (t \in \mathbf{R}) \tag{2.23}$$

$$C_{\beta,0} = \beta \tag{2.24}$$

$$D_{\alpha,\beta} = \alpha^2/2\beta^2 \tag{2.25}$$

In particular, we have

$$R_{\alpha,\beta}(t) = \frac{\alpha}{2(2\pi)^{1/2}\beta} E_{\alpha,\beta}(t) \qquad [t \in (0, \infty)] \tag{2.26}$$

which corresponds to (d) in Theorem 2.2(ii). Therefore, we can rewrite relation (i) in Theorem 2.1 as

$$\frac{1}{\beta - i\zeta} = \frac{1}{R_{\alpha,\beta}(0)} \int_0^\infty e^{i\zeta t} R_{\alpha,\beta}(t)\, dt \qquad [\zeta \in \mathbf{C}^+ \cup (\mathbf{R} - \{0\})] \tag{2.27}$$

From (2.21) and (2.25), we immediately obtain

$$\alpha^2/2 = R_{\alpha,\beta}(0)\beta \tag{2.28}$$

$$D_{\alpha,\beta} = R_{\alpha,\beta}(0)/\beta \tag{2.29}$$

which corresponds to (ii) and (iv) in Theorem 2.1 together with (2.24), respectively.

Concerning the physical meaning of Theorem 2.1, we give three remarks.

Remark 2.1. The left-hand side in relation (i) in Theorem 2.1 [resp. relation (2.27)] denotes the complex mobility of the system X described by Eq. (2.11) [resp. Eq. (2.20)], which represents the response of the system described by Eq. (2.11) [resp. Eq. (2.20)] to the external force. On the other hand, the right-hand side in relation (i) in Theorem 2.1 [resp. relation (2.27)] is determined by the outer function (resp. covariance function) of the system X and so the spectral measure of X, which represents the thermal fluctuation of the system X in equilibrium without the external force.

Furthermore, the left-hand side in relation (ii) in Theorem 2.1 [resp. relation (2.28)] denotes the fluctuation power of the random force $\alpha\dot{B}$ in Eq. (2.11) [resp. Eq. (2.20)]. On the other hand, the right-hand side in relation (ii) in Theorem 2.1 [resp. relation (2.28)] is determined by the positive constants $R(0)$ and $C_{\beta,\gamma}$ [resp. $R(0)$ and β]. The positive constant $C_{\beta,\gamma}$ (resp. β) is determined by the drift coefficient representing the systematic part of Eq. (2.11) [resp. Eq. (2.20)]. It is physically allowable for us to regard the positive constant $R(0)$ as the absolute constant kT in equilibrium described by Eq. (2.11) [resp. Eq. (2.20)], where k and T denote the Boltzmann constant and absolute temperature, respectively.

The relation (2.29) was first discovered by Nyquist,[20] who showed that the random electromotive force appearing across a resistor is determined by its impedance. In this case, the response to external force is represented by the dissipation of the energy.

In order to distinguish two kinds of representations that relate, for a system in thermal equilibrium, two physically distinct quantities of fundamental experimental significance—the fluctuation behavior and the dissipative behavior[10]—Kubo[14–16] called relations (2.27) and (2.28) the *first fluctuation-dissipation theorem* and the *second fluctuation-dissipation theorem*, respectively. By taking this into account, we call relations (i) and (ii) in Theorem 2.1 the *generalized first fluctuation-dissipation theorem* and the *generalized second fluctuation-dissipation theorem*, respectively. Furthermore, we call the positive constant $C_{\beta,\gamma}$ the *generalized friction constant.*

Remark 2.2. We note[14–16] that the positive constant D defined by (2.14) is transformed as

$$D = \lim_{t \to \infty} \frac{E([\int_0^t X(s)\,ds]^2)}{2t} \tag{2.30}$$

This is called the *diffusion constant* or the fluctuation power of X. Relation (2.29), which indicates that the diffusion constant D is inversely proportional to the friction coefficient β in Eq. (2.20), is called the *Einstein relation.*[8,14–16] Further,[14–16] the Einstein relation (2.29) corresponds to a special case of the first fluctuation-dissipation theorem (2.27).

However, we find from relation (iv) in Theorem 2.1 that there occurs in the system described by Eq. (2.11) a deviation from the Einstein relation (2.29) for the system described by Eq. (2.20), with the degree of deviation calculated by formula (v) in Theorem 2.1. Furthermore, we will find in Section 7 that the degree of deviation from the Einstein relation can be concretely parametrized for the Stokes–Boussinesq–Langevin equation with white noise as the random foce, which gives a concrete and physical example of the first KMO-Langevin equation (2.11). For this reason, we call relation (iv) the *generalized Einstein relation.*

Remark 2.3. We note that relation (iii) in Theorem 2.1 for the system described by Eq. (2.11) is the same as relation (2.25) for the system described by Eq. (2.20).

3. A GENERALIZED FLUCTUATION-DISSIPATION THEOREM. 2

In this section, we treat the case where the measure σ in (2.1) satisfies the following conditions:

$$\int_0^\infty \lambda^{-1}\sigma(d\lambda) < \infty \tag{3.1}$$

$$\int_0^\infty \lambda\sigma(d\lambda) = \infty \tag{3.2}$$

Corresponding to (2.8), by Theorems 5.2 and 6.1 in Ref. 24, we obtain the *first KMO-Langevin data* $(\beta, \rho) \in \mathscr{L}_\infty$ associated with σ such that for any $\zeta \in \mathbf{C}^+$

$$h(\zeta) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\beta - i\zeta \int_0^\infty [1/(\lambda - i\zeta)]\, \rho(d\lambda)} \tag{3.3}$$

Furthermore, we know from Proposition 7.1 in Ref. 24 that relation (3.3) implies that X satisfies, in the sense of random tempered distributions,

$$-\beta\mathbf{X} - \lim_{\varepsilon \downarrow 0} \gamma_\varepsilon * \dot{\mathbf{X}} + \dot{\mathbf{B}} = 0 \tag{3.4}$$

Similar to (2.13), we define the *generalized friction constant* $C_{\beta,\gamma}^\infty$ by

$$C_{\beta,\gamma}^\infty = \pi \left[\int_{\mathbf{R}} \left| \beta - i\xi \lim_{\varepsilon \downarrow 0} \int_0^\infty e^{i\xi t}\, \gamma_\varepsilon(t)\, dt \right|^{-2} d\xi \right]^{-1} \tag{3.5}$$

and the diffussion constant D by (2.14). Corresponding to (ii)–(v) in Theorem 2.1, we shall show the following:

Proposition 3.1:

(i) $\frac{1}{2} = R(0)\, C^{\infty}_{\beta,\gamma}$

(ii) $D = 1/2\beta^2$

(iii) $D = \dfrac{R(0)}{\beta} \dfrac{C^{\infty}_{\beta,\gamma}}{\beta}$

(iv) $\dfrac{C^{\infty}_{\beta,\gamma}}{\beta} = \displaystyle\int_0^{\infty} \frac{R(t)}{R(0)}\, \gamma(t)\, dt$

Proof. Similar to (ii) in Theorem 2.1, we have (i) from (H.4) in Ref. 24 and (3.3). By noting (2.16), we have (ii) from formula (i) in Theorem 6.3 in Ref. 24. Part (iii) follows immediately from (i) and (ii). For the proof of (iv), we claim that for almost all $t > 0$,

$$-\beta \int_0^t E(s)\, ds - (\gamma * E)(t) + (2\pi)^{1/2} = 0 \tag{3.6}$$

$$\beta \int_0^t R(s)\, ds + \int_0^{\infty} R(t-s)\, \gamma(s)\, ds = \frac{1}{2\beta} \tag{3.7}$$

By noting that $E \in L^1 \cap L^2_{\mathrm{loc}}$ and then using Eq. (3.4), we find that (3.6) can be proved similarly as (2.17). We note that (3.6) implies, for any $t > 0$,

$$\begin{aligned}(2\pi)^{1/2} \int_0^{\infty} E(s)\, ds = \beta \int_0^{\infty} E(s) \left[\int_0^{s+t} E(u)\, du \right] ds \\ + \int_0^{\infty} E(s)(\gamma * E)(s+t)\, ds\end{aligned}$$

Furthermore, since it folows from Lemma 5.2(iv) and Theorem 5.1(iv) in Ref. 24 and (ii) that

$$\int_0^{\infty} E(s)\, ds = (2\pi)^{1/2}/\beta$$

we find that (3.7) can be proved similarly as (2.19). In particular, we see from (3.7) that

$$\int_0^{\infty} R(s)\, \gamma(s)\, ds = 1/2\beta$$

which together with (i) gives (v). ∎

Since we cannot regard Eq. (3.4) as an equation describing the time evolution of X, it is not clear how (3.3), (i), and (iii) in Proposition 3.1 give the generalized first fluctuation-dissipation theorem, generalized second fluctuation-dissipation theorem, and generalized Einstein relation, respectively, on the basis of Eq. (3.4). In order to derive an equation that describes the time evolution of X, we used in Ref. 24 the following condition:

$$\int_0^\infty \lambda^{-2}\, \sigma(d\lambda) < \infty \tag{3.8}$$

We know from Theorem 7.1 in Ref. 24 that Eq. (3.4) can be rewritten as the following *first KMO-Langevin equation*:

$$\dot{\mathbf{X}} = -\frac{\beta}{q}\mathbf{X} + \frac{1}{q}(Q \underset{s}{*} \dot{\mathbf{X}}) + \frac{1}{q}\dot{\mathbf{B}} \tag{3.9}$$

where q and Q are a positive constant and a bounded measurable function on $\mathbf{R}$, respectively, defined by

$$q = \int_0^\infty \gamma(s)\, ds \tag{3.10}$$

$$Q(t) = \chi_{[0,\infty)}(t) \int_0^\infty \gamma(s)\, ds \tag{3.11}$$

Finally, we shall show how (3.3), (i), (iii), and (iv) in Proposition 3.1 give the generalized first fluctuation-dissipation theorem, generalized second fluctuation-dissipation theorem, generalized Einstein relation, and deviation from the Einstein relation, respectively, on the basis of Eq. (3.9).

Theorem 3.1.:

(i) For any $\zeta \in \mathbf{C}^+$

$$\left[\frac{\beta}{q} + (-i\zeta) + (-i\zeta)^2 \int_0^\infty e^{i\zeta t}\frac{Q(t)}{q}\, dt\right]^{-1} = \left[\lim_{\varepsilon \downarrow 0} \frac{h(i\varepsilon)^{-1} - [\pi^{-1}\int_0^\infty R(t)\, dt]^{-1}}{\varepsilon}\right] h(\zeta)$$

(ii) $$\frac{(1/q)^2}{2} = R(0)\, C^\infty_{\beta/q, \gamma/q}$$

(iii) $$D = \frac{(1/q)^2}{2(\beta/q)^2}$$

(iv) $$D = \frac{R(0)}{\beta/q} \frac{C^{\infty}_{\beta/q, \gamma/q}}{\beta/q}$$

(v) $$1 - \frac{C^{\infty}_{\beta/q, \gamma/q}}{\beta/q} = \frac{1}{qR(0)} \int_0^{\infty} \left(1 - \frac{R(t)}{R(0)}\right) \gamma(t)\, dt$$

Proof. By (3.10) and (3.11), we can rewrite (3.3) as

$$h(\zeta) = \frac{1/q}{(2\pi)^{1/2}} \frac{1}{\beta/q + (-i\zeta) + (-i\zeta)^2 \int_0^{\infty} e^{i\zeta t}\, Q(t)\, dt}$$

for any $\zeta \in \mathbf{C}^+$. On the other hand, it follows from (2.14), (3.3), Proposition 3.1(ii), and (3.10) that

$$(2\pi)^{1/2} q = \lim_{\varepsilon \downarrow 0} \frac{h(i\varepsilon)^{-1} - [\pi^{-1} \int_0^{\infty} R(t)\, dt]^{-1}}{\varepsilon}$$

and so we have (i). Since $C^{\infty}_{\beta,\gamma}$ defined by (3.5) satisfies

$$C^{\infty}_{\beta/q, \gamma/q} = \frac{1}{q^2} C^{\infty}_{\beta,\gamma} \tag{3.12}$$

we see that (ii) follows immediately from Proposition 3.1(i). Similarly, Proposition 3.1(ii) and Proposition 3.1(iii) with (3.12) give (iii) and (iv), respectively. Since

$$\frac{C^{\infty}_{\beta/q, \gamma/q}}{\beta/q} = \frac{1}{\beta/q} C^{\infty}_{\beta,\gamma}$$

we find that (v) follows from Proposition 3.1(iv) and (3.10). ∎

Remark 3.1. Since (i), (ii), and (v) in Theorem 3.1 play the same role for Eq. (3.11) as (i), (ii), and (iv) in Theorem 2.1 do for Eq. (2.11), it is reasonable for us to call (i), (ii), and (iv) in Theorem 3.1 the *generalized first fluctuation-dissipation theorem*, the *generalized second fluctuation-dissipation theorem*, and the *generalized Einstein relation*, respectively.

Remark 3.2. We note that (v) in Theorem 3.1 implies that the degree of deviation from the Einstein relation is less than 1, different from (v) in Theorem 2.1, which implies that it is greater than or equal to 1.

Remark 3.3. Since we treat the case where condition (3.2) is satisfied, we find that X has no Markovian property and so the generalized friction constant $C^{\infty}_{\beta,\gamma}$ cannot give such a characterization of Markovian property for X as Theorem 2.2(ii).

4. KUBO'S FLUCTUATION-DISSIPATION THEOREM

In this section, we treat the case where the measure σ in (2.1) satisfies condition (2.6), but does not necessarily satisfy condition (2.7):

$$\int_0^\infty \lambda^{-1}\,\sigma(d\lambda) < \infty \tag{4.1}$$

Together with the first KMO-Langevin data (α, β, ρ) in (2.8) or (β, ρ) in (3.3), by Theorems 8.1 and 8.5 in Ref. 24, we have the *second KMO-Langevin data* $(\alpha_0, \beta_0, \rho_0) \in \mathscr{L}_1$ associated with σ such that for any $\zeta \in \mathbf{C}^+$

$$\frac{1}{2\pi}\int_0^\infty e^{i\zeta t}R(t)\,dt = \frac{\alpha_0}{(2\pi)^{1/2}}\left[\beta_0 - i\zeta - i\zeta\int_0^\infty [1/(\lambda - i\zeta)]\,\rho_0(d\lambda)\right]^{-1} \tag{4.2}$$

We have found[24] that it is impossible to derive an equation with white noise as a random force that describes the time evolution of X. For this reason, we have in Ref. 24 introduced the colored noise $\mathbf{I} = (I(\phi); \phi \in \mathscr{S}(\mathbf{R}))$, which is called the *Kubo noise*, as a stationary random tempered distribution:

$$I(\phi) = \int_{\mathbf{R}} \left(\frac{h}{h_0}\hat{\phi}\right)^{\sim}(t)\,dB(t) \tag{4.3}$$

where h_0 is an L^2-function on $\mathbf{R}$ defined by

$$h_0 = (\chi_{(0,\infty)}R)^{\sim} \tag{4.4}$$

By Theorem 8.3 in Ref. 24, we have the following representation with the causal property:

$$\int_{\mathbf{R}} X(t)\,\phi(t)\,dt = \frac{1}{(2\pi)^{1/2}}\int_0^\infty R(t)\,I(\phi(\cdot + t))\,dt \qquad [\phi \in \mathscr{S}(\mathbf{R})] \tag{4.5}$$

$$\mathscr{F}_{\mathbf{X}}^{-}(t) = \mathscr{F}_{\mathbf{I}}^{-}(t) \qquad (t \in \mathbf{R}) \tag{4.6}$$

which gives a mathematical justification of the random force in Kubo's linear response theory.[13-16] By using (4.2) and (4.3), we know from Theorem 8.4 in Ref. 24 that the time evolution of X, in the sense of random tempered distributions, can be governed by the following *second KMO-Langevin equation*:

$$\dot{\mathbf{X}} = -\beta_0\mathbf{X} - \lim_{\varepsilon \downarrow 0} \dot{\gamma}_{0,\varepsilon} * \mathbf{X} + \alpha_0\mathbf{I} \tag{4.7}$$

where for any $\varepsilon \geqslant 0$, $\gamma_{0,\varepsilon}$ and γ_0 are functions on $\mathbf{R}$ defined by

$$\gamma_{0,\varepsilon}(t) = \chi_{(0,\infty)}(t) \int_{\varepsilon}^{\infty} e^{-t\lambda} \rho(d\lambda) \tag{4.8}$$

$$\gamma_0 = \gamma_{0,0} \tag{4.9}$$

We further define the tempered distribution R_{γ_0} by

$$R_{\gamma_0}(\phi) = \int_{\mathbf{R}} \phi(t) \operatorname{sgn}(t) \gamma_0(t)\, dt \qquad [\phi \in \mathscr{S}(\mathbf{R})] \tag{4.10}$$

Then, corresponding to Theorem 2.1 or Theorem 3.1, we show the following:

Theorem 4.1:

(i) For any $\zeta \in \mathbf{C}^+$

$$\left[\beta_0 - i\zeta - i\zeta \lim_{\varepsilon \downarrow 0} \int_0^{\infty} e^{i\zeta t} \gamma_{0,\varepsilon}(t)\, dt\right]^{-1} = \frac{1}{R(0)} \int_0^{\infty} e^{i\zeta t} R(t)\, dt$$

(iia) $$\Delta_{\mathrm{W}}(d\xi) = \frac{R(0)}{\pi}\left[\beta_0 + \int_0^{\infty} \frac{\xi^2}{\lambda^2 + \xi^2} \rho_0(d\lambda)\right] d\xi$$

$$= \frac{R(0)}{\pi} \operatorname{Re}\left[\beta_0 - i\xi \lim_{\varepsilon \downarrow 0} \int_0^{\infty} e^{i\xi t} \gamma_{0,\varepsilon}(t)\, dt\right] d\xi$$

(iib) $$R_{\mathrm{W}} = \frac{R(0)}{\pi}(\beta_0 \delta + \dot{R}_{\gamma_0})$$

where Δ_{W} and R_{W} are the spectral measure and the covariance distribution of the random tempered distribution $\mathbf{W} = \alpha_0 \mathbf{I}$.

(iii) $D = R(0)/\beta_0$

Proof. We recall formula (i) in Theorem 8.2 in Ref. 24:

$$\alpha_0 = R(0)/(2\pi)^{1/2} \tag{4.11}$$

Equation (4.2) together with (4.11) gives (i). Part (ii) follows from Propositons 8.1(iii) and (iv) in Ref. 24. Finally, (iii) follows from formula (ii) in Theorem 8.2 in Ref. 24. ∎

Remark 4.1. By using (4.11), we find that (i) and (ii) in Theorem 8.1 correspond to the *first fluctuation-dissipation theorem* and the *second fluctuation-dissipation theorem*, respectively, in the sense of

Remark 2.1. Furthermore, (iii) is the *Einstein relation.* We call (i)–(iii) in Theorem 4.1 *Kubo's fluctuation-dissipation theorem.*

Remark 4.2. By (iii) in Theorem 4.1 and (4.11), we obtain

$$D = (2\pi)^{1/2}\alpha_0/\beta_0 \tag{4.12}$$

which implies that relation (2.25) for the system described by Eq. (2.20) does not hold for the system described by the second KMO-Langevin equation (4.7), in contrast to relation (iii) in Theorem 2.1 [resp. relation (iii) in Theorem 3.1] for the system described by the first KMO-Langevin equation (2.11) [resp. the first KMO-Langevin equation (3.9)].

Remark 4.3. However, we find from formulas (i) and (ii) in Theorem 3.3 in Ref. 24 and the reasoning in the proof of Theorem 8.2 in Ref. 24 that

$$\alpha_0^2/2\beta_0^2 = D_{X_0} \tag{4.13}$$

where the positive constant D_{X_0} is the diffusion constant of the stationary Gaussian process X_0 with the covariance function R_0 given by

$$R_0(t) = \frac{1}{2\pi}\int_0^\infty R(|t|+s)\,R(s)\,ds \qquad (t \in \mathbf{R}) \tag{4.14}$$

5. STOKES–BOUSSINESQ–LANGEVIN EQUATION

We shall consider the motion of a sphere of radius r and mass m moving with an arbitrary velocity $X(t)$ at time t in a fluid with viscosity η and density ρ. Denoting by $W = (W(t);\, t \in \mathbf{R})$ and $F = (F(t);\, t \in \mathbf{R})$ the fluctuating force and drag force acting on the sphere, respectively, we see that Newton's equation becomes

$$m\, dX(t)/dt = -F(t) + W(t) \qquad \text{in } \mathbf{R} \tag{5.1}$$

By taking the inverse Fourier transform of both sides of Eq. (5.1), we find

$$m(-i\xi)\,\tilde{X}(\xi) = -\tilde{F}(\xi) + \tilde{W}(\xi) \qquad \text{in } \mathbf{R} \tag{5.2}$$

and so for fixed almost all $\xi \in \mathbf{R} - \{0\}$

$$m\frac{d}{dt}\,[e^{-it\xi}\tilde{X}(\xi)] = -e^{-it\xi}\tilde{F}(\xi) + e^{-it\xi}\tilde{W}(\xi) \qquad \text{in } \mathbf{R} \tag{5.3}$$

Equation (5.3) represents the time evolution of the motion of a sphere of radius r and mass m vibrating with frequency ξ in such a situation that the velocity, fluctuating force, and drag force at time t are given by $e^{-it\xi}\tilde{X}(\xi)$, $e^{-it\xi}\tilde{W}(\xi)$, and $e^{-it\xi}\tilde{F}(\xi)$, respectively. By solving a linearized Navier–Stokes equation subject to imcompressibility and stick boundary conditions in hydrodynamics, Stokes[28] showed that the drag force for Eq. (5.3) is given by

$$e^{-it\xi}\tilde{F}(\xi)=6\pi r\eta\left[1+r\left(\frac{\xi\rho}{2\eta}\right)^{1/2}\right]e^{-it\xi}\tilde{X}(\xi)$$
$$+3\pi r^2\left(\frac{2\rho\eta}{\xi}\right)^{1/2}\left[1+\frac{2r}{9}\left(\frac{\xi\rho}{2\eta}\right)^{1/2}\right]\frac{d}{dt}\left[e^{-it\xi}\tilde{X}(\xi)\right]$$

and so

$$\tilde{F}(\xi)=\left\{(6\pi r\eta+3\pi r^2 2\rho\eta\xi)\right.$$
$$\left.+\left[3\pi r^2(2\rho\eta)^{1/2}\frac{1}{\sqrt{\xi}}+\frac{2\pi r^3}{3}\rho\right](-i\xi)\right\}\tilde{X}(\xi) \qquad (5.4)$$

By taking the Fourier transform of both sides of Eq. (5.4), Boussinesq[5,6,18] has shown that the drag force F for Eq. (5.1) is given by

$$F(t)=2\pi\rho r^3\left[\frac{1}{3}\frac{dX(t)}{dt}+\frac{3\eta}{r^2\rho}X(t)\right.$$
$$\left.+\frac{3}{r}\left(\frac{\eta}{\rho\pi}\right)^{1/2}\int_{-\infty}^{t}\frac{1}{(t-s)^{1/2}}\frac{dX(s)}{ds}ds\right]$$

Therefore we find that Eq. (5.1) can be rewritten as

$$m^*\frac{dX(t)}{dt}=-6\pi r\eta X(t)-6\pi r^2\left(\frac{\rho\eta}{\pi}\right)^{1/2}\int_{-\infty}^{t}\frac{1}{(t-s)^{1/2}}\frac{dX(s)}{ds}ds+W(t) \qquad (5.5)$$

where m^* is the effective mass given by

$$m^*=m+\tfrac{2}{3}\pi r^3\rho \qquad (5.6)$$

Definition 5.1. We call Eq. (5.5) the *Stokes–Boussinesq–Langevin equation.*

From the viewpoint of the theory of stochastic differential equations, it seems necessary to pay attention to the second term on the right-hand side of Eq. (5.5), because it is a sort of singular integral. For this reason, we return to (5.4) and then substitute it into (5.2) to obtain

$$\tilde{X}(\xi)=[(2\pi)^{1/2}h_S(\xi)]\,\tilde{W}(\xi) \qquad (5.7)$$

where h_S is the frequency response function on $\mathbf{C}^{+\cup}(\mathbf{R}-\{0\})$ defined by

$$h_S(\zeta)=\frac{1}{(2\pi)^{1/2}}\frac{1}{6\pi r\eta+m^*(-i\zeta)+6\pi r^2(\rho\eta)^{1/2}(-i\zeta)^{1/2}} \tag{5.8}$$

where for any complex $\zeta\in\mathbf{C}-\{z\in\mathbf{C};\ \operatorname{Re} z\leqslant 0,\ \operatorname{Im} z=0\}$ $(-i\zeta)^{1/2}$ stands for $\exp\frac{1}{2}(\log|\zeta|+i\operatorname{Arg}\zeta)$ $(-\pi<\operatorname{Arg}\zeta<\pi)$.

By using the formula

$$\int_0^\infty \frac{1}{\lambda-i\zeta}\frac{1}{\lambda^{1/2}}\,d\lambda=\frac{\pi}{(-i\zeta)^{1/2}} \qquad \text{for any} \quad \zeta\in\mathbf{C}^+ \tag{5.9}$$

we find that the function h_S in (5.8) can be rewritten as

$$h_S(\zeta)=\frac{\alpha_{\mathrm{SB}}}{(2\pi)^{1/2}}\frac{1}{\beta_{\mathrm{SB}}-i\zeta-i\zeta\int_0^\infty[1/(\lambda-i\zeta)]\,\rho_{\mathrm{SB}}(d\lambda)} \tag{5.10}$$

where α_{SB} and β_{SB} are positive constants and ρ_{SB} a Borel measure on $[0,\infty)$ given by

$$\alpha_{\mathrm{SB}}=/m^* \tag{5.11}$$

$$\beta_{\mathrm{SB}}=6\pi r\eta/m^* \tag{5.12}$$

$$\rho_{\mathrm{SB}}(d\lambda)=\frac{6r^2(\rho\eta)^{1/2}}{m^*}\frac{1}{\lambda^{1/2}}\,d\lambda \tag{5.13}$$

By applying Theorems 3.1 and 8.5 in Ref. 24 tp the triple $(\alpha_{\mathrm{SB}},\beta_{\mathrm{SB}},\rho_{\mathrm{SB}})$, we obtain two Borel measures σ_{SB} and v_{SB} on $[0,\infty)$ such that

$$\sigma_{\mathrm{SB}}=L_1^{-1}((\alpha_{\mathrm{SB}},\beta_{\mathrm{SB}},\rho_{\mathrm{SB}})) \tag{5.14}$$

$$v_{\mathrm{SB}}=L_2^{-1}((\alpha_{\mathrm{SB}},\beta_{\mathrm{SB}},\rho_{\mathrm{SB}})) \tag{5.15}$$

In fact, it follows from Theorem 2.1 and Lemma 2.6 in Ref. 24 that

$$\sigma_{\mathrm{SB}}(d\lambda)=\frac{1}{2\pi}\left[\int_0^\infty\frac{1}{\lambda+\lambda'}\,v_{\mathrm{SB}}(d\lambda')\right]v_{\mathrm{SB}}(d\lambda) \tag{5.16}$$

$$E_S(t)=\hat{h}_S(t)=\chi_{[0,\infty)}(t)\int_0^\infty e^{-t\lambda}v_{\mathrm{SB}}(d\lambda) \tag{5.17}$$

The concrete form of the measure v_{SB} in (5.15) and (5.17) is obtained in Refs. 11 and 17:

$$v_{\mathrm{SB}}(d\lambda)=\left(\frac{2}{\pi}\right)^{1/2}\alpha_{\mathrm{SB}}e_{\mathrm{SB}}\frac{\lambda^{1/2}}{(\beta_{\mathrm{SB}}-\lambda)^2+e_{\mathrm{SB}}^2\lambda}\,d\lambda \tag{5.18}$$

where e_{SB} is a positive constant defined by

$$e_{SB} = 6\pi r^2(\rho\eta)^{1/2}/m \tag{5.19}$$

Furthermore, we see from (5.9), (5.10), (5.17), and (5.18) that the measure σ_{SB} in (5.14) and (5.16) can be represented by

$$\sigma_{SB}(d\lambda) = \frac{\alpha_{SB}^2}{\pi} \frac{e_{SB}}{\beta_{SB} + \lambda + e_{SB}\sqrt{\lambda}} \frac{\sqrt{\lambda}}{(\beta_{SB} - \lambda)^2 + e_{SB}^2 \lambda}\, d\lambda \tag{5.20}$$

We note that

$$\int_0^\infty \lambda^{-2}\sigma_{SB}(d\lambda) = \int_0^\infty \lambda^{-2}\nu_{SB}(d\lambda) = \infty \tag{5.21}$$

$$\int_0^\infty (1+\lambda^{-1})\,\sigma_{SB}(d\lambda) + \int_0^\infty (1+\lambda^{-1})\,\nu_{SB}(d\lambda) < \infty \tag{5.22}$$

$$\int_0^\infty \lambda\nu_{SB}(d\lambda) = \infty \tag{5.23}$$

$$\int_0^\infty \lambda\sigma_{SB}(d\lambda) < \infty \tag{5.24}$$

$$\int_0^\infty \lambda^2\sigma_{SB}(d\lambda) = \infty \tag{5.25}$$

6. THE SECOND STOKES–BOUSSINESQ–LANGEVIN EQUATION

In this section we consider the nonnegative-definite function R_K on $\mathbf{R}$ defined by

$$R_K(t) = \int_0^\infty e^{-|t|\lambda}\nu_{SB}(d\lambda) \tag{6.1}$$

where ν_{SB} is the Borel measure on $[0, \infty)$ in (5.18). Let $X_K = (X_K(t);\, t \in \mathbf{R})$ be a real, stationary Gaussian process on a probability space $(\Omega, \mathscr{F}, P)$ with mean zero and R_K its correlation function, and with h_K the outer function of R_K [see (H.1) in Ref. 24].

In order to derive an equation that describes the time evolution of X_K, we use the Kubo noise $\mathbf{I} = (I(\phi);\, \phi \in \mathscr{S}(\mathbf{R}))$, which was defined by (4.3) as a stationary random tempered distribution:

$$I(\phi)(\omega) = \int_{\mathbf{R}} \left(\frac{h_k}{h_S}\hat{\phi}\right)^{\sim}(t)\, dB(t, \omega), \qquad \text{a.s. } \omega \tag{6.2}$$

where $(B(t);\ t \in \mathbf{R})$ is a one-dimensional Brownian motion. Since (5.15) implies that the triple $(\alpha_{SB}, \beta_{SB}, \rho_{SB})$ represents the second KMO-Langevin data associated with ν_{SB} having condition (5.22), we know from (4.7) that the time evolution of X_K is governed by the following *second KMO-Langevin equation*:

$$\dot{\mathbf{X}}_K = -\beta_{SB}\mathbf{X}_K - \lim_{\varepsilon \downarrow 0} \gamma_{SB,\varepsilon} * \dot{\mathbf{X}}_K + \alpha_{SB}\mathbf{I} \tag{6.3}$$

in the sense of random tempered distributions, where for each $\varepsilon > 0$, $\gamma_{SB,\varepsilon}$ is a function defined on $\mathbf{R}$ by

$$\gamma_{SB,\varepsilon}(t) = \chi_{(0,\infty)}(t) \int_{\varepsilon}^{\infty} e^{-t\lambda} \rho_{SB}(d\lambda) \tag{6.4}$$

We have

$$\lim_{\varepsilon \downarrow 0} \gamma_{SB,\varepsilon}(t) = \chi_{(0,\infty)}(t) \frac{6\sqrt{\pi r^2 \rho\eta}}{m^*} \frac{1}{\sqrt{t}} \qquad \text{for any} \quad t \in \mathbf{R} \tag{6.5}$$

We find that the second KMO-Langevin equation (6.3) gives a realization for the Stokes–Boussinesq–Langevin equation (5.5) that is characterized by a qualitative nature of T-positivity. However, we find from the results of Section 7 in Ref. 24 that the time evolution of X_K cannot be described by the first KMO-Langevin equation which has white noise as the random force, because of conditions (5.21) and (5.23). We call Eq. (6.3) the *second Stokes–Boussinesq–Langevin equation*. From Theorem 4.1(iii), we have the *Einstein relation*:

$$D_K = R_K(0)/\beta_{SB} \tag{6.6}$$

where D_K is the diffusion constant of X_K, given by

$$D_K = \int_0^{\infty} R_K(t)\, dt \tag{6.7}$$

Furthermore, it follows from (4.11) and (6.3) that

$$R_K(0) = (2\pi)^{1/2} \alpha_{SB} \tag{6.8}$$

and so

$$D_K = \frac{1}{3(2\pi)^{1/2}} \frac{1}{r\eta} \tag{6.9}$$

Concerning the long-time tail behavior of the correlation function R_K, we know from Refs. 11 and 17 that the following *Alder–Wainwright effect* holds:

$$\lim_{t \to \infty} (\beta_{SB} t)^{3/2} R_K(t) = \frac{R_K(0)}{2\sqrt{\pi}} a \tag{6.10}$$

where a is a positive constant defined by

$$a = e_{SB}/\beta_{SB}^{1/2} \tag{6.11}$$

From (5.6), (5.11), (5.12), (6.8), and (6.11), we have

$$a = \left[\frac{6\pi r^3 \rho}{m + (2/3)\pi r^3 \rho}\right]^{1/2} \tag{6.12}$$

$$\frac{R_K(0)}{2\sqrt{\pi}} a = \frac{3\pi r^3 \rho}{[m + (2/3)\pi r^3 \rho]^3} \tag{6.13}$$

7. THE FIRST STOKES–BOUSSINESQ–LANGEVIN EQUATION

In this section we consider the nonnegative-definite function R_W on $\mathbf{R}$ defined by

$$R_W(t) = \alpha_W^2 \int_0^\infty e^{-|t|\lambda} \sigma_{SB}(d\lambda) \tag{7.1}$$

where α_W is a fixed positive constant and σ_{SB} is the Borel measure on $[0, \infty)$ in (5.20). Let $X_W = (X_W(t);\ t \in \mathbf{R})$ be a real, stationary Gaussian process with mean zero and R_W as its correlation function, and with h_W the outer function of R_W [see (H.1) in Ref. 24]. Then we note that the realization of X_W can be given by

$$X_W(t) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbf{R}} E_W(t-s)\, dB(s) \tag{7.2}$$

where $E_W = \hat{h}_W$ and $(B(t);\ t \in \mathbf{R})$ is the one-dimensional Brownian motion in (6.2). Furthermore, we find from (5.14)–(5.17) that

$$h_W = \alpha_W h_S \tag{7.3}$$

$$E_W = \alpha_W E_S \tag{7.4}$$

Since (5.14) implies that the triple $(\alpha_W \alpha_{SB}, \beta_{SB}, \rho_{SB})$ represents the first KMO-Langevin data associated with $\alpha_W \sigma_{SB}$ satisfying conditions (5.22) and (5.24), we know from (2.11) that the time evolution of X_W is governed by the following *first KMO-Langevin equation*:

$$\dot{\mathbf{X}}_W = -\beta_{\mathrm{SB}}\mathbf{X}_W - \lim_{\varepsilon \downarrow 0} \gamma_{\mathrm{SB},\varepsilon} * \dot{\mathbf{X}}_W + \alpha_W \alpha_{\mathrm{SB}} \dot{\mathbf{B}} \tag{7.5}$$

in the sense of random tempered distributions.

As with the second Stokes–Boussinesq–Langevin equation (6.3), we find that Eq. (7.5) gives the realization for the Stokes–Boussinesq–Langevin equation (5.5), which is also characterized by a qualitative nature of T-positivity. Though the time evolution of X_W can be described by a second KMO-Langevin equation having a Kubo noise as a random force, the coefficients in the systematic part of its equation become different from those in the systematic part of the Stokes–Boussinesq–Langevin equation (5.5), as we have seen in Sections 2–4. We call Eq. (7.5) the *first Stokes–Boussinesq–Langevin equation.*

From (ii) and (iv) in Theorem 2.1, we have the following *generalized second fluctuation-dissipation theorem* and *generalized Einstein relation*, respectively:

$$\frac{(\alpha_W \alpha_{\mathrm{SB}})^2}{2} = R_W(0)\, C_{\mathrm{SB}} \tag{7.6}$$

$$D_W = \frac{R_W(0)}{\beta_{\mathrm{SB}}} \frac{C_{\mathrm{SB}}}{\beta_{\mathrm{SB}}} \tag{7.7}$$

where C_{SB} is the *generalized friction constant* in the systematic part of the first Stokes–Boussinesq–Langevin equation (7.5) and D_W is the diffusion constant of X_W:

$$C_{\mathrm{SB}} = \pi \left\{ \int_{\mathbf{R}} |\beta_{\mathrm{SB}} - i\xi[1 + e_{\mathrm{SB}}(-i\xi)^{1/2}]|^{-2}\, d\xi \right\}^{-1} \tag{7.8}$$

$$D_W = \int_0^\infty R_W(t)\, dt \tag{7.9}$$

We shall now investigate the value $C_{\mathrm{SB}}/\beta_{\mathrm{SB}}$ in relation (7.7), which gives the degree of deviation from the Einstein relation. By (5.20) and (7.1), we get

$$R_W(0) = \frac{(\alpha_W \alpha_{\mathrm{SB}})^2 a^3}{e_{\mathrm{SB}}^2} \int_0^\infty \frac{1}{1 + y + a\sqrt{y}} \frac{\sqrt{y}}{(1-y)^2 + a^2 y}\, dy \tag{7.10}$$

which, together with the generalized Einstein relation (7.6), gives

$$\frac{C_{\mathrm{SB}}}{\beta_{\mathrm{SB}}} = \frac{\pi}{2a} \left[\int_0^\infty \frac{1}{1 + y + a\sqrt{y}} \frac{\sqrt{y}}{(1-y)^2 + a^2 y}\, dy \right]^{-1} \tag{7.11}$$

Furthermore, it follows from (5.9), (5.20), and formula (v) in Theorem 2.1 that

$$\frac{C_{\mathrm{SB}}}{\beta_{\mathrm{SB}}}-1=\frac{(\alpha_W\alpha_{\mathrm{SB}})^2a^3}{R_W(0)\,\beta_{\mathrm{SB}}e_{\mathrm{SB}}}\int_0^\infty\frac{1}{1+y+a\sqrt{y}}\,\frac{1}{(1-y)^2+a^2y}\,dy$$

and so by (7.10)

$$\frac{C_{\mathrm{SB}}}{\beta_{\mathrm{SB}}}-1=a\,\frac{\int_0^\infty(1+y+a\sqrt{y})^{-1}[(1-y)^2+a^2y]^{-1}\,dy}{\int_0^\infty(1+y+a\sqrt{y})^{-1}\sqrt{y}/[(1-y)^2+a^2y]\,dy} \tag{7.12}$$

Next, we show that the Alder–Wainwright effect also holds for the correlation function R_W. Since by (5.16), (5.17), (6.1), (7.1), and (7.4)

$$R_W(t)=\frac{\alpha_W^2}{2\pi}\int_0^\infty R_{\mathrm{K}}(t+s)\,R_{\mathrm{K}}(s)\,ds\qquad\text{for any}\quad t>0$$

we see from the bounded convergence theorem, (6.6), (6.8), and the Alder–Wainwright effect (6.10) for the correlation function R_{K} that

$$\begin{aligned}\lim_{t\to\infty}(\beta_{\mathrm{SB}}t)^{3/2}R_W(t)&=\frac{\alpha_W^2}{2\pi}\,\frac{R_{\mathrm{K}}(0)}{2\sqrt{\pi}}\,aD_{\mathrm{K}}\\&=\frac{(\alpha_W\alpha_{\mathrm{SB}})^2}{2}\,\frac{a}{2\sqrt{\pi}\beta_{\mathrm{SB}}}\end{aligned}$$

which, together with the generalized Einstein relation (7.6) and (7.11), yields the following *Alder–Wainwright effect*:

$$\lim_{t\to\infty}(\beta_{\mathrm{SB}}t)^{3/2}R_W(t)=\frac{\sqrt{\pi}}{2}R_W(0)\left(\int_0^\infty\frac{1}{1+y+a\sqrt{y}}\,\frac{\sqrt{y}}{(1-y)^2+a^2y}\,dy\right)^{-1} \tag{7.13}$$

Finally, we calculate the diffusion constant D_W of X_W. From the generalized Einstein relation (7.7), (7.10), and (7.11), we get

$$D_W=\frac{(\alpha_W\alpha_{\mathrm{SB}})^2a^2}{\beta_{\mathrm{SB}}e_{\mathrm{SB}}^2}=\frac{1}{2\beta_{\mathrm{SB}}}\left(\frac{\alpha_W\alpha_{\mathrm{SB}}a}{e_{\mathrm{SB}}}\right)^2$$

and so by (5.15), (5.16), and (6.11)

$$D_W=\tfrac{1}{2}(\alpha_W/6\pi r\eta)^2 \tag{7.14}$$

8. CONVERGENCE TO ORNSTEIN–UHLENBECK BROWNIAN MOTION

It is known in statistical physics[16] that

$$a = 3(1 + 2\,\rho_0/\rho)^{-1/2} \tag{8.1}$$

where ρ_0 and ρ are the density of a Brownian particle and a fluid in the Stokes–Boussinesq–Langevin equation (5.5), respectively. Furthermore, it is also known that if ρ_0 is far bigger than ρ, that is, ρ is extremely small, then a Brownian particle moves according to the Langevin equation without delayed drift term.

In this section, we show that the processes X_{K} and X_W have an asymptotic behavior when a tends to zero for fixed η that is equivalent to the $\lim \rho \to 0$ behavior for fixed η, by noting (6.12).

By (5.8), we have

$$\lim_{a \to 0} h_S(\xi) = h_{1/m, \beta_\infty}(\xi) \qquad \text{for any} \quad \xi \in \mathbf{R} \tag{8.2}$$

$$|h_S(\xi)|^2 \leqslant \frac{1}{[(2\pi)^{1/2} m]^2} \frac{1}{\beta_\infty^2 + \xi^2} \qquad \text{for any} \quad \xi \in \mathbf{R} \tag{8.3}$$

$$0 \leqslant \operatorname{Re} h_S(\xi) \leqslant \frac{1}{2\pi m} \frac{\beta_{\mathrm{SB}}^2 + a\beta_{\mathrm{SB}}^{1/2} + (|\xi|/2)^{1/2}}{\beta_\infty^2 + \xi^2} \qquad \text{for any} \quad \xi \in \mathbf{R} \tag{8.4}$$

where β_∞ is a positive constant given by

$$\beta_\infty = 6\pi r \eta / m \tag{8.5}$$

Since it follows from (E.2) in Ref. 24, (5.17), and (6.1) that

$$R_{\mathrm{K}}(t) = 2(\operatorname{Re} h_S)^\wedge(t) \qquad (t \in \mathbf{R}) \tag{8.6}$$

we see from (8.2) and (8.4) that

$$\lim_{a \to 0} R_{\mathrm{K}}(t) = R_{\alpha_\infty, \beta_\infty}(t) = \frac{(2\pi)^{1/2}}{m} e^{-\beta_\infty |t|} \qquad (t \in \mathbf{R}) \tag{8.7}$$

where α_∞ is a positive constant given by

$$\alpha_\infty = [2(2\pi)^{1/2} (\beta_\infty / m)]^{1/2} \tag{8.8}$$

On the other hand, since it follows from (H.4) in Ref. 24 and (7.3) that

$$R_W(t) = \alpha_W^2 (|h_S|^2)^\wedge(t) \qquad (t \in \mathbf{R}) \tag{8.9}$$

we see from (8.2) and (8.8) that

$$\lim_{a\to 0} R_W(t) = R_{\alpha_W/m,\beta_\chi}(t) = \frac{(\alpha_W/m)^2}{2\beta_\chi} e^{-\beta_\chi|t|} \qquad (t\in\mathbf{R}) \tag{8.10}$$

We can conclude from (8.7) and (8.10) that

$$\lim_{a\to 0} X_{\mathrm{K}} = X_{\alpha_\chi,\beta_\chi} \qquad \text{in law} \tag{8.11}$$

$$\lim_{a\to 0} X_W = X_{\alpha_W/m,\beta_\chi} \qquad \text{in law} \tag{8.12}$$

In particular, we find that in order that X_{K} and X_W have the same limit process (the Ornstein–Uhlenbeck Brownian motion $X_{\alpha_\chi,\beta_\chi}$) as a tends to zero, it is a necessary and sufficient condition that

$$\alpha_W = (12\sqrt{2}\pi^{3/2} r\eta)^{1/2} \tag{8.13}$$

It then follows from (7.14) that under condition (8.13) the diffusion constant D_W becomes

$$D_W = \frac{1}{3(2\pi)^{1/2}} \frac{1}{\eta} \tag{8.14}$$

which coincides with the diffusion constants D_{K} and $D_{X_{\alpha_\chi,\beta_\chi}}$ of the processes X_{K} and $X_{\alpha_\chi,\beta_\chi}$, given by (2.25) and (6.9), respectively.

Next, we investigate the $\lim a\to 0$ behavior of the value $C_{\mathrm{SB}}/\beta_{\mathrm{SB}}$ in relation (7.7). By (7.12), we have

$$\begin{aligned}
\frac{C_{\mathrm{SB}}}{\beta_{\mathrm{SB}}} - 1 &= \sqrt{a}\,\frac{\int_{\sqrt{a}}^{\infty} (1+y+a\sqrt{y})^{-1}\sqrt{a}/[(1-y)^2+a^2y]\,dy}{\int_0^\infty (1+y+a\sqrt{y})^{-1}\sqrt{y}/[(1-y)^2+a^2y]\,dy} \\
&\quad + a\,\frac{\int_0^{\sqrt{a}} (1+y+a\sqrt{y})^{-1}[(1-y)^2+a^2y]^{-1}\,dy}{\int_0^\infty (1+y+a\sqrt{y})^{-1}\sqrt{y}/[(1-y)^2+a^2y]\,dy} \\
&\leqslant \sqrt{a} + \frac{a\int_0^{\sqrt{a}} [1/(1-y)^2]\,dy}{\int_2^\infty (1+y+a\sqrt{y})^{-1}\sqrt{y}/[(1-y)^2+a^2y]\,dy}
\end{aligned}$$

and so, by letting a tend to zero, we have

$$\lim_{a\to 0} \frac{C_{\mathrm{SB}}}{\beta_{\mathrm{SB}}} = 1$$

which, together with (8.10), implies that the generalized Einstein relation (7.7) approaches the Einstein relation for the Ornstein–Uhlenbeck–Brownian motion $X_{\alpha_\infty, \beta_\infty}$.

Finally, we investigate the double limits $t \to \infty$ and $a \to 0$ of the ratio between the correlation functions R_K and R_W. From (8.7) and (8.10), we immediately have

$$\lim_{t \to \infty} \left(\lim_{a \to 0} \frac{R_W(t)}{R_K(t)} \right) = \frac{\alpha_W^2}{12\sqrt{2}\pi^{3/2} r\eta}$$

On the other hand, we see from (7.11) and the Alder–Wainwright effect (6.10) and (7.13) for the correlation functions R_K and R_W that

$$\begin{aligned} \lim_{t \to \infty} \frac{R_W(t)}{R_K(t)} &= \lim_{t \to \infty} \frac{(\beta_{SB} t)^{3/2} R_W(t)}{(\beta_{SB} t)^{3/2} R_K(t)} \\ &= 2 \frac{C_{SB} R_W(0)}{\beta_{SB} R_K(0)} \end{aligned}$$

and so, by (8.7), (8.10), and (8.15),

$$\lim_{a \to 0} \left(\lim_{t \to \infty} \frac{R_W(t)}{R_K(t)} \right) = \frac{\alpha_W^2}{6\sqrt{2}\pi^{3/2} r\eta}$$

Therefore, by (8.13), we have the following interesting limit theorem:

$$\lim_{a \to 0} \left(\lim_{t \to \infty} \frac{R_W(t)}{R_K(t)} \right) = 2 \tag{8.16}$$

$$\lim_{t \to \infty} \left(\lim_{a \to 0} \frac{R_W(t)}{R_K(t)} \right) = 1 \tag{8.17}$$

ACKNOWLEDGMENTS

I would like to express my gratitude to Profs. S. Albeverio, Ph. Blanchard, and L. Streit for the kind invitation to the BiBoS Research Center at Bielefeld University, where part of this paper was written.

REFERENCES

1. B. J. Alder and T. E. Wainwright, *Phys. Rev. Lett.* **18**:988–990 (1967).
2. B. J. Alder and T. E. Wainwright, *Phys. Rev. A* **1**:18–21 (1970).
3. J. Bosse, W. Götze, and M. Lücke, *Phys. Rev. A* **20**:1603–1607 (1979).
4. A. Bouiller, J.-P. Boon, and P. Deguent, *J. Phys. (Paris)* **39**:159–165 (1978).

5. J. Boussinesq, *Comptes Rendus* **100**:935–937 (1885).
6. J. Boussinesq, *Théorie analytique de la chaleur* (Gauthier-Villars, Paris, 1903).
7. J. L. Doob, *Ann. Math.* **43**:351–369 (1942).
8. A. Einstein, *Drude's Ann.* **17**:549–560 (1905).
9. P. D. Fedele and Y. W. Kim, *Phys. Rev. Lett.* **44**:691–694 (1980).
10. D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, Reading, Massachusetts, 1975).
11. E. H. Hauge and A. Martin-Löf, *J. Stat. Phys.* **7**:259–281 (1973).
12. Y. W. Kim and J. E. Matta, *Phys. Rev. Lett.* **31**:208–211 (1973).
13. R. Kubo, *J. Phys. Soc. Japan* **12**:570–586 (1957).
14. R. Kubo, *1965 Tokyo Summer Lectures in Theoretical Physics*, Part I. *Many-Body Theory* (Benjamin, New York, 1966), pp. 1–16.
15. R. Kubo, *Rep. Prog. Phys.* **29**:255–284 (1966).
16. R. Kubo, *Statistical Physics, Foundations of Modern Physics 5* (Iwanami-Shoten, Tokyo, 1972) (in Japanese).
17. R. Kubo, RIMS, Kyoto, October, 1979, pp. 50–93 (in Japanese).
18. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, Massachusetts, 1959).
19. P. Langevin, *Comptes Rendus* **146**:530–533 (1908).
20. H. Nyquist, *Phys. Rev.* **32**:110–113 (1928).
21. Y. Okabe, *J. Fac. Sci. Univ. Tokyo IA* **26**:115–165 (1979).
22. Y. Okabe, *J. Fac. Sci. Univ. Tokyo IA* **28**:169–213 (1981).
23. Y. Okabe, *Commun. Math. Phys.* **98**:449–468 (1985).
24. Y. Okabe, *J. Fac. Sci. Univ. Tokyo IA* **33**:1–56 (1986).
25. Y. Okabe, in preparation.
26. K. Oobayashi, T . Kohno, and H. Utiyama, *Phys. Rev. A* **27**:2632–2641 (1983).
27. G. L. Paul and P. N. Pussey, *J. Phys. A: Math. Gen.* **14**:3301–3327 (1981).
28. G. G. Stokes, *Mathematical and Physical Papers* (1966), Vol. 3, pp. 1–141 [reprinted from *Trans. Camb. Phil. Soc.* **9** (1850)].
29. G. E. Uhlenbeck and L. S. Ornstein, *Phys. Rev.* **36**:823–841 (1930).
30. A. Widom, *Phys. Rev. A* **3**:1394–1396 (1971).
31. R. Zwanzig and M. Bixon, *Phys. Rev. A* **2**:2005–2012 (1970).